Multivariate Saddle integrals, 5.1, 5.2, and 5.3 based on Analytic Combinatorics in Several Variables by Robin Pemantle and Mark C. Wilson

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February 20, 2013

1 Review of single variable saddle integrals

Recall that Steve showed us:

$$\int_\gamma A(z)e^{-\lambda\phi(z)}dz$$

is asymptotic to

$$A(z_0)\sqrt{\frac{2\pi}{\phi''(z_0)\lambda}}e^{-\lambda\phi(z_0)}$$

and the first few terms in the expansion near the origin as $\lambda \to \infty$. Remember the proof where the first few coefficients were obtained via analytic inversion, and a mistake was found by Steve regarding the exponent of the big-Oh term.

2 Overview of 5.1

We continue with this set up with A as our amplitude and ϕ as the phase, both analytic functions, but this time of a vector argument $\mathbf{z}$ along the contour C, a d-chain in $\mathbb{C}^d$. Compared to the one variable case where Theorem 4.1.1 covers all degrees of degeneracy of the phase function ϕ ($k \geq 2$), and all degrees of vanishing of the amplitude function A ($l \geq 0$), for the multivariate case ϕ has a much greater range of possibilities.

Recall that in one dimension, we take $k = 2$; for higher dimensions, we assume the *Hessian matrix*

$$\mathcal{H} := \left(\frac{\partial^2\phi}{\partial z_j \partial z_k}\right) \neq 0.$$

The Taylor series for ϕ expanded around the origin is

$$\phi(\mathbf{z}) = \phi(\mathbf{0}) + \mathbf{z}^T\nabla\phi(\mathbf{0}) + \frac{1}{2}\mathbf{z}^T\mathcal{H}\mathbf{z} + O(|\mathbf{z}|^3),$$

hence the Hessian matrix represents twice the quadratic term in the phase, and its non-singularity is a generalization of non-vanishing of the quadratic term in the univariate case.

Instead of the special phase function x^2, we will use $S(\mathbf{x}) = x_1^2 + \cdots + x_d^2$ to denote the standard quadratic. Parallel to the development of the univariate case, we will establish the result

$$A = \text{monomial} \qquad\qquad \phi = \text{standard quadratic}$$

coupled with a big-Oh result which allows us to integrate term by term to obtain asymptotics for the standard phase function.

Three main theorems:

Theorem 1 (5.1.1 Standard Phase)**.** *Let $A(\mathbf{x})$ be a real analytic function defined on a neighbourhood $\mathcal{N}$ of the origin in $\mathbb{R}^d$ with a series expansion*

$$A(\mathbf{x}) := \sum_{r_1,\ldots,r_d} x_1^{r_1} \cdots x_d^{r_d} = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{x}^{\mathbf{r}}.$$

Let

$$I(\lambda) := \int_{\mathcal{N}} A(\mathbf{x}) e^{-\lambda S(\mathbf{x})} d\mathbf{x}.$$

Then an asymptotic series expansion for $I(\lambda)$ in increasing $|\mathbf{r}|$ is

$$I(\lambda) \sim \sum_n \sum_{|\mathbf{r}|=n} a_{\mathbf{r}} \beta_{\mathbf{r}} \lambda^{-(|r|+d)/2}$$

where $\beta_{\mathbf{r}} = 0$ if any r_j is odd, and

$$\beta_{2m} = \sqrt{\pi}^d \cdot \prod_{j=1}^{d} \frac{(2m_j)!}{m_j! 4^{m_j}},$$

otherwise.

Theorem 2 (5.1.2 Re(ϕ) has a strict minimum)**.** *Suppose that the real part of ϕ is strictly positive except at the origin and that its Hessian matrix $\mathcal{H}$ is non-singular there. Let A be any analytic function not vanishing at the origin and define*

$$I(\lambda) := \int_{\mathcal{N}} A(z) e^{-\lambda \phi(z)} dz.$$

Then

$$I(\lambda) \sim \sum_{l \geq 0} c_l \lambda^{-d/2-l},$$

where

$$c_0 = A(0) \cdot \frac{\sqrt{2\pi}^{-d}}{\sqrt{\det(\mathcal{H})}},$$

and the choice of sign is defined by taking the product of the principal square roots of the eigenvalues of $\mathcal{H}$.

Theorem 3 (5.4.8 Critical point decomposition for stratified spaces)**.** *Let A and ϕ be analytic functions on a neighbourhood of a stratified space $\mathcal{M} \subseteq \mathbb{C}^d$. If ϕ has finitely many critical points on $\mathcal{M}$, then*

$$I(\lambda) \sim (2\pi\lambda)^{-d/2} \sum_{\mathbf{x}} A(\mathbf{x}) e^{\lambda\phi(\mathbf{x})} \det(\mathcal{H}(\mathbf{x}))^{-1/2}$$

where

$$\mathcal{H}(\mathbf{x}) \text{ is the Hessian for } \phi \text{ at } \mathbf{x},$$

and the sum is over the critical points $\mathbf{x}$ at which the real part of ϕ is minimized.

3 5.2 Standard phase

Remember how Steve developed the single variate case by starting at the simplest case:

$$A = \text{monomial} \qquad \text{and} \qquad \phi = x^2.$$

We will begin with a proposition which evaluates a real integral exactly.

Proposition 4 (5.2.1)**.** *The integral*

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx = \beta_{2n} = \sqrt{\pi} \cdot \frac{(2n)!}{n! 4^n}.$$

Note that the exponent of the monomial A is $2n$, and the exponent of the monomial and monic ϕ is 2.

Proof. We will prove this proposition by induction.

The **basis step** is when $n = 0$. This is, up to a change of variables and observation of symmetry, the standard Gaussian integral and is in fact the definition of $\Gamma(1/2)$ – which is $\sqrt{\pi}$. This can be checked directly using the substitution $u = x^2$ in the integral.

The **inductive step** is to assume the result for $n - 1$. We use integration by parts to get

$$\begin{aligned}
\int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx &= \int_{-\infty}^{\infty} \frac{x^{2n-1}}{-2} (-2xe^{-x^2}) dx \\
&= \frac{-x^{2n-1}}{2} \cdot e^{-x^2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{(2n-1)x^{2n-2}}{-2} e^{-x^2} dx \\
&= 0 + \frac{2n-1}{2} \sqrt{\pi} \cdot \frac{(2n-2)!}{(n-1)! 4^{n-1}},
\end{aligned}$$

by the inductive hypothesis, and the result follows from multiplying and dividing the expression by $2n$.

Since the result for $n - 1$ implies the result for n, by mathematical induction we have shown that the result of the proposition holds. □

Now we can vary the phase function ϕ so that it is no longer monic, but has a factor of λ. This is stated in the next Corollary.

Corollary 5 (5.2.2)**.**

$$\int_{-\infty}^{\infty} x^{2n} e^{-\lambda x^2} dx = \beta_{2n} \lambda^{-1/2-n}.$$

Proof. We just need a change of variables $y = \sqrt{\lambda} x$. This implies $dy = \sqrt{\lambda} dx$ and thus

$$\int_{-\infty}^{\infty} x^{2n} e^{-\lambda x^2} dx = \int_{-\infty}^{\infty} \frac{1}{\lambda^n \cdot \sqrt{\lambda}} y^{2n} e^{-y^2} dx = \lambda^{-n-1/2} \beta_{2n}.$$

□

Corollary 6 (5.2.3 Higher dimensional monomial integral)**.** *Let* $\mathbf{r}$ *be a d-vector of nonnegative integers. Then*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{r_1} x_2^{r_2} \ldots x_d^{r_d} e^{-\lambda(x_1^2 + x_2^2 + \ldots + x_d^2)} dx_1 dx_2 \ldots dx_d = \prod_{j=1}^{d} \beta_{r_j} \cdot \lambda^{-(d+|\mathbf{r}|)/2},$$

where $\beta_{r_j} = 0$ *if* r_j *is odd (and thus the integral is nonzero only when each* r_j *is even).*

Proof. When our integral is written out as a d-dimensional integral, you can see how integrating each dimension separately implies the integral has the value

$$\prod_{j=1}^{d} \left(\int_{-\infty}^{\infty} x_j^{r_j} e^{-\lambda x_j^2} dx_j \right) = \prod_{j=1}^{d} \beta_{r_j} \lambda^{-(1+r_j)/2}$$
$$= \prod_{j=1}^{d} \beta_{r_j} \cdot \lambda^{-(d+|\mathbf{r}|)/2}$$

□

Proposition 7 (5.2.4 big-O estimate)**.** *Let* A *be any smooth function satisfying a big-O bound at the origin*

$$A(\mathbf{x}) = O(|\mathbf{x}|^r)$$

where the norm is the Euclidean norm, and r *is just some positive real number, not a vector as in previous corollary. Then the integral over any connected compact set* K *containing the origin may be bounded from above by*

$$\int_K A(\mathbf{x}) e^{-\lambda S(\mathbf{x})} dx = O(\lambda^{-(d+r)/2}).$$

The implied constant on the right goes to zero as the constant in the hypothesis of the upper bound goes to zero.

Proof. 1. Because K contains the origin, is connected, compact, and $A(\mathbf{x}) = O(|\mathbf{x}|^r)$ at the origin, there exists a constant c such that $|A(\mathbf{x})| \leq c|\mathbf{x}|^r$ in K.

2. Let us create a sequence of sets that are intersections of K with either the ball

$$K_0 := \{\mathbf{x} : |\mathbf{x}| \leq \lambda^{-1/2}\}$$

or the shells

$$K_n := K \cap \{2^{n-1}\lambda^{-1/2} \leq |\mathbf{x}| \leq 2^n\lambda^{-1/2}\}.$$

These sets help us say more precisely how $|A(\mathbf{x})|$ is bounded.

3. We can also bound

$$\int_{K_0} e^{-\lambda S(\mathbf{x})} d\mathbf{x} \leq \int_{K_0} d\mathbf{x} \leq c_d \lambda^{-d/2},$$

for some constant c_d. Thus, combining the previous points gives

$$\left| \int_{K_0} A(\mathbf{x}) e^{-\lambda S(\mathbf{x})} d\mathbf{x} \right| \leq c' \lambda^{-(r+d)/2}.$$

4. For $n \geq 1$, on K_n we use A's big-O bound to obtain

$$|A(\mathbf{x})| = O(|\mathbf{x}|^r) \leq 2^{rn} \cdot c \cdot \lambda^{-r/2}.$$

5. We can use our bound on $|\mathbf{x}|$ between the shells to give us a bound on $|\mathbf{x}|^2$

$$2^{2n-2}/\lambda \leq |\mathbf{x}|^2 \leq 2^n/\lambda.$$

Thus,

$$e^{-\lambda S(\mathbf{x})} \leq e^{-2^{2n-2}}.$$

6. Finally, the integral bound in K_n is

$$\int_{K_n} d\mathbf{x} \leq 2^{dn} c_d \lambda^{-d/2}.$$

7. Combining the last three bounds, we have the bound for the entire integral by summing over all the shells. Let

$$c'' = c \cdot c_d \sum_{n=1}^{\infty} 2^{(d+r)n} e^{-2^{2n-2}} < \infty.$$

Then

$$\int_K A(\mathbf{x}) e^{-\lambda S(\mathbf{x})} d\mathbf{x} = \sum_{k=0}^{\infty} \left| \int_{K_n} A(\mathbf{x}) e^{-\lambda S(\mathbf{x})} d\mathbf{x} \right| \leq (c' + c'')\lambda^{-(r+d)/2}.$$

□

These four propositions and corollaries make it easier to construct the proof of Theorem 5.1.1. (Standard Phase).

Proof of Theorem 5.1.1. Write $A(\mathbf{x})$ as a power series up to degree N plus a remainder term:

$$A(\mathbf{x}) = \sum_{n=0}^{N} \left(\sum_{|\mathbf{r}|=n} a_{\mathbf{r}} x^{\mathbf{r}} \right) + R(\mathbf{x}),$$

where $R(\mathbf{x}) = O(|\mathbf{x}|^{N+1})$.

Now we have a monomial part of A, along with a big-O estimate. Using Corollary 5.2.3 on the monomial integral and Proposition 5.2.4 on the big-O estimate thus implies the desired result:

$$I(\lambda) = \sum_{n=0}^{N} \left(\sum_{|\mathbf{r}|=n} a_{\mathbf{r}} \beta_{\mathbf{r}} \lambda^{-(n+d)/2} \right) + O(\lambda^{-(N+1+d)/2}).$$

□

4 5.3 Real part of phase has a strict minimum

Here, we have the set up:

1. Let $\mathcal{N}$ be a neighbourhood of the origin in $\mathbb{R}^d$.
2. We have an analytic $\phi : \mathcal{N} \to \mathbb{C}^d$ which is represented by a power series that converges on $\mathcal{N}$.
3. Such a ϕ may be extended to a holomorphic function on a neighbourhood $\mathcal{N}_{\mathbb{C}}$ of the origin in complex d-dim space.
4. Now, suppose $\phi(\mathbf{0}) = 0$ and the real part of ϕ is non-negative on $\mathcal{N}$. This section's assumption that the real part of phase ϕ has a strict minimum implies that the gradient of ϕ must vanish at the origin.
5. We say that ϕ has a quadratically non-degenerate critical point at the origin if the quadratic part of ϕ is non-degenerate.
6. Recall in the expansion of ϕ where the quadratic part of ϕ is a quadratic form represented by
$$\frac{1}{2} z^T \mathcal{H} z$$
7. Non-degeneracy of a quadratic form means non-singularity of the Hessian $\mathcal{H}$; the determinant of a quadratic form means the determinant of $\mathcal{H}$.

8. Review of Hessian behaviour under a change of variables: If $\psi : \mathbb{C}^d \to \mathbb{C}^d$ is a bi-holomorphic map, $\nabla\phi(\psi(\mathbf{y})) = 0$ when $\psi(y) = x$, and the Hessian matrix $\mathcal{H}$ exists there, then the Hessian matrix $\mathcal{H}'$ of the composed map $\phi \circ \psi$ at $\mathbf{y}$ is given by

$$\mathcal{H}' = J_\psi^T \mathcal{H} J_\psi$$

where J_ψ is the Jacobian matrix of the map ψ at $\mathbf{y}$:

$$J_\psi = \begin{pmatrix} \frac{\partial \psi_1}{\partial y_1} & \cdots & \frac{\partial \psi_1}{\partial y_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_d}{\partial y_1} & \cdots & \frac{\partial \psi_d}{\partial y_d} \end{pmatrix}.$$

We need two lemmas before the proof of Theorem 5.1.2 (Re(ϕ) has a strict minimum). The first lemma reassures us that near the origin, if ψ is not of our standard quadratic form $S(\mathbf{x})$ then we can find local coordinates to change $\mathbf{x}$ into $\mathbf{y}$ where a standard quadratic form is attained. The second lemma provides the equivalence between verifying the sign choice of a composed derivative in the multivariate case and a determinantal condition.

Lemma 8 (5.3.1)**.** *There is a bi-holomorphic change of variables* $\mathbf{x} = \psi(\mathbf{y})$ *such that*

$$\phi(\psi(\mathbf{y})) = S(\mathbf{y}) = y_1^2 + \cdots + y_d^2.$$

The differential

$$J_\psi = d\psi(0) \qquad \textit{satisfies} \quad (\det J_\psi)^2 = \frac{1}{\det(\mathcal{H}(\phi)/2)}.$$

Recall what Nicolas taught us about Morse theory – this lemma is the Morse Lemma.

Proof. Let us do the easy part first: consider

$$\tilde{\mathcal{H}}(S) = J_\psi^T \mathcal{H}(\phi) J_\psi.$$

Compute the Hessian of the standard quadratic form S to get $\tilde{\mathcal{H}}(S) = 2I$, where I is the identity matrix. Then

$$1 = \det(\tilde{\mathcal{H}}(S)/2) = \det\left(J_\psi^T \frac{\mathcal{H}(\phi)}{2} J_\psi\right) = \det(J_\psi)^2 \cdot \det(\mathcal{H}(\phi)/2),$$

and thus

$$(\det J_\psi)^2 = \frac{1}{\det(\mathcal{H}(\phi)/2)}.$$

The long part is the change of variables where we break the part into three steps.

Step 1 Rewrite $\phi(\mathbf{x})$ as an expansion in coordinates $x_j x_k$ multiplied by the entries of $\mathcal{H}$.

Step 2 Use mathematical induction to morph the y_j's one at a time into the standard quadratic form by assuming that none of the diagonal entries of the Hessian is 0.

Step 3 Take care of the case when some diagonal entry of the Hessian is 0 by using a unitary conjugation.

□

Lemma 9 (5.3.2)**.** *Let $W \subseteq \mathbb{C}^d$ be the set $\{\mathbf{z} : \mathrm{Re}(S(\mathbf{z})) > 0\}$. Pick any $\alpha \in \mathrm{GL}_d(\mathbb{C})$ mapping $\mathbb{R}^d$ into $\overline{W}$, and let $M := \alpha^\dagger \alpha$ be the matrix representing $S \circ \alpha$. Let $\pi : \mathbb{C}^d \to \mathbb{R}^d$ be the projection onto the real part. Then $\pi \circ \alpha$ is orientation preserving on $\mathbb{R}^d$ iff $\det \alpha$ is the product of the principal square roots of the eigenvalues of M.*

Proof. We will need lots of linear algebra in this proof.

First suppose $\alpha \in \mathrm{GL}_d(\mathbb{R})$. Then $M := \alpha^T \alpha$ is Hermition and thus has an eigen-decomposition $M = P^{-1} D P$. As $zMz^T = (z\alpha^T)(z\alpha^T)^T = |z\alpha^T|^2 \geq 0$ for all z, we see that $yDy^T \geq 0$ by a change of variables. As D is a diagonal matix whose entries are the eigenvalues of M, these eigenvalues are positive. Therefore, the product of their principal square roots is positive.

The map π is the identity on $\mathbb{R}^d$, so an equivalent statement would be: The linear transformation α preserves orientation iff it has positive determinant. (This is true by definition).

In general, define $\alpha_t := \pi_t \circ \alpha$, where

$$\pi_t(\mathbf{z}) = \Re\{\mathbf{z}\} + (1-t)i\Im\{\mathbf{z}\}.$$

This should remind us of the homotopic map Nicolas showed us last semester.

For all $0 \leq t \leq 1$,

$$\pi_t(\mathbb{R}^d) \subseteq \overline{W},$$

so $M_t := \alpha_t^T \alpha_t$ has eigenvalues with nonnegative real parts.

The product of the principal square roots of the eigenvalues is a continuous function on the set of non-singular matrices with no negative real eigenvalues. The determinant of α_t is a continuous function of t, and when $t = 1$ we have seen that it agrees with the product of principal square roots of eigenvalues of M_t; thus by continuity, this is the correct sign choice for all $0 \leq t \leq 1$. We take $t = 0$ to prove the lemma. □

Proof of Theorem 5.1.2: Re(ϕ) has a strict minimum. The power series we got from Theorem 5.1.1 allows us to extend ϕ to a neighbourhood of the origin in $\mathbb{C}^d$.

Using Lemma 5.3.1, we can apply the change of variables ψ to turn a random ϕ into a standard quadratic:

$$I(\lambda) = \int_{\psi^{-1}\mathcal{C}} A \circ \psi(\mathbf{y}) e^{-\lambda S(\mathbf{y})} \det(d\psi(\mathbf{y})) d\mathbf{y} = \int_{\psi^{-1}\mathcal{C}} \tilde{A}(\mathbf{y}) e^{-\lambda S(\mathbf{y})} d\mathbf{y},$$

where $\mathcal{C}$ is a neighbourhood of the origin in $\mathbb{R}^d$ with the standard orientation.

Here, we need to check that we can move the chain $\psi^{-1}\mathcal{C}$ of integration back to the real plane. If successful, then Theorem 5.1.1 can be used to yield the desired expansion in powers $\lambda^{-(d/2+l)}$.

Take the real part of $S(\mathbf{z})$ and call it $h(\mathbf{z})$. The chain $\mathcal{C}' := \psi^{-1}(\mathcal{C})$ lies in the region $\{\mathbf{z} \in \mathbb{C}^d : h(\mathbf{z}) > 0\}$ except when $\mathbf{z} = \mathbf{0}$, and in particular $h \geq \epsilon > 0$ on $\partial\mathcal{C}'$.

Next, we will define a homotopy from the identity map to the map π projecting out the imaginary part of $\mathbf{z}$. For any chain σ where the integration takes place, this homotopy induces a chain homotopy supported on the image of the support of σ under the homotopy.

Let

$$H(\mathbf{z}, t) := \Re\{\mathbf{z}\} + (1-t)i\Im\{\mathbf{z}\}.$$

Then $H(\sigma)$ is a chain homotopy satisfying

$$\partial H(\sigma) = \sigma - \pi\sigma + H(\partial\sigma).$$

With $\sigma = C'$, in addition to observing that $S(H(\mathbf{z}, t)) \geq S(\mathbf{z})$, we see there is a $d+1$-chain $\mathcal{D}$ with

$$\partial\mathcal{D} = C' - \pi C' + C''$$

and $\mathcal{C}''$ supported on $\{h > \epsilon\}$.

Recall Stokes' Theorem: $\int \partial_D w = \int_D dw = 0$ when w is a holomorphic d-form. Here we use $\partial_D = C' - \pi C' + C''$ to obtain

$$\int_{C'} w = \int_{\pi C'} w - \int_{C''} w.$$

Taking $w = \tilde{A}e^{-\lambda S}$ and noting $\int_{C''} w = O(e^{-\lambda\epsilon})$, this tells us

$$I(\lambda) = \int_{\pi C'} \tilde{A}(\mathbf{y})e^{-\lambda S(\mathbf{y})}d\mathbf{y} + O(e^{-\lambda\epsilon}).$$

The integral in the above expression is looked after by the last lemma. □